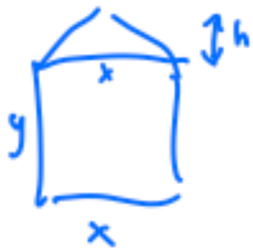
2.48



Fixed Perimeter

Maximize Area.

Constraint: $P(x,y,h) = 2y + x + 2\sqrt{h^2+\left(\frac{x}{2}\right)^2} = P_0$

$x, y, h \geq 0$, $x \leq P_0/2$, $y \leq P_0/2$, $h \leq P_0/2$

Function to maximize: $A(x,y,h) = xy + \frac{1}{2}xh$

$$\nabla A = \lambda \nabla P$$

$P(x,y,h) = P_0$ fixed perimeter.

$$\left(y+\frac{h}{2},\ x,\ \frac{x}{2}\right) = \lambda\left(1+\frac{x}{2\sqrt{\frac{x^2}{4}+h^2}},\ 2,\ \frac{2h}{\sqrt{\frac{x^2}{4}+h^2}}\right)$$

$$2y + x + 2\sqrt{h^2+\left(\frac{x}{2}\right)^2} = P_0 .$$

$$y+\frac{h}{2} = \lambda\left(1+\frac{x}{2\sqrt{\frac{x^2}{4}+h^2}}\right)$$

$$x = 2\lambda$$

$$\frac{x}{2} = \lambda\frac{2h}{\sqrt{\frac{x^2}{4}+h^2}}$$

$\lambda = \frac{x}{2}$ $\rightarrow$

$$y+\frac{h}{2} = \frac{x}{2}\left(1+\frac{x}{2\sqrt{\frac{x^2}{4}+h^2}}\right)$$

$$\frac{x}{2} = \frac{x}{2}\frac{2h}{\sqrt{\frac{x^2}{4}+h^2}}$$

As long as $x \neq 0$, $\frac{2h}{\sqrt{\frac{x^2}{4}+h^2}} = 1 \Rightarrow$

$$2h = \sqrt{\frac{x^2}{4}+h^2}$$

$$4h^2 = \frac{x^2}{4}+h^2$$

$$3h^2 = \frac{x^2}{4} \Rightarrow x^2 = 12h^2$$

$$\Rightarrow x = \sqrt{\underset{4\cdot 3}{12}}\,h$$

$$\Rightarrow \boxed{x = 2\sqrt{3}\,h}$$

$$y + \frac{h}{2} = \sqrt{3}h\left(1 + \frac{\sqrt{3}h}{\sqrt{\frac{x^2}{4}+h^2}}\right)$$

$$\frac{x^2}{4} = \frac{12h^2}{4} = 3h^2$$

$$\sqrt{4h^2} = 2h$$

$$y + \frac{h}{2} = \sqrt{3}h\left(1 + \frac{\sqrt{3}}{2}\right) = \sqrt{3}h + \frac{3}{2}h$$

$$y = \sqrt{3}h + h = (\sqrt{3}+1)h$$

$$2y + x + 2\sqrt{h^2 + \left(\frac{x}{2}\right)^2} = P_0$$

$$2(\sqrt{3}+1)h + 2\sqrt{3}h + 2\cdot 2h = P_0$$

$$h(2\sqrt{3}+2+2\sqrt{3}+4) = P_0$$

$$h(4\sqrt{3}+6) = P_0$$

$$\boxed{h = \frac{P_0}{4\sqrt{3}+6},\quad x = \frac{2\sqrt{3}P_0}{4\sqrt{3}+6},\quad y = \frac{(\sqrt{3}+1)P_0}{4\sqrt{3}+6}.}$$

(only critical point, and $A(x,y,h) \to 0$ on boundary — so this must be the global max!.

Partial derivatives of $P(x,y,h)$:

$$\left(2y + x + 2\sqrt{h^2 + \underbrace{\left(\tfrac{x}{2}\right)^2}_{x^2/4}}\right)_x$$

$$= 1 + 2\cdot\frac{1}{2}\left(h^2 + \frac{x^2}{4}\right)^{-1/2}\cdot\frac{2x}{4}$$

$$= 1 + \frac{x}{2\sqrt{h^2 + x^2/4}}$$

$$(\;''\;)_h = 2\cdot\frac{1}{2}\left(h^2 + \frac{x^2}{4}\right)^{-1/2}\cdot 2h$$

$$= \frac{2h}{\sqrt{h^2 + x^2/4}}.$$

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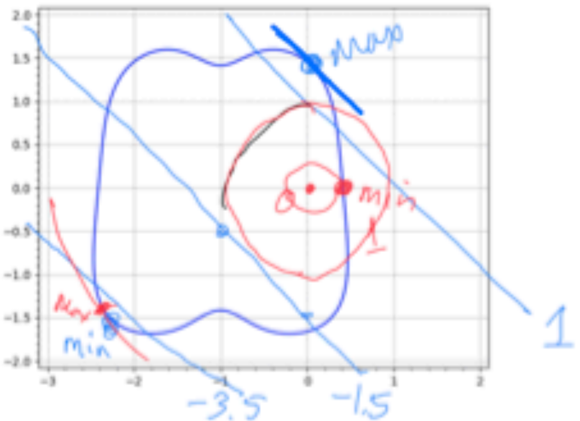
6. Find the absolute maximum of the function $F(x_1, x_2) = e^{x_1 - x_2}$ when restricted to the ellipse $x_2^2 + x_1 x_2 + 4x_1^2 = 9$.

7. Consider the graph of $G(x, y) = 5$ below.



$(x^2+y^2)^{3/2} = C$

$x^2+y^2 = \underbrace{C^{2/3}}_{R^2}$

(a) Estimate the locations of the absolute maximum and minimum of the function $F(x, y) = x + y$ on $G(x, y) = 5$. look at $x+y=$ constant.

(b) Estimate the locations of the global maximum and minimum of the function $C(x, y) = (x^2 + y^2)^{3/2}$ on $G(x, y) = 5$. look at $(x^2+y^2)^{3/2} =$ const.

8. A container of that should hold 1000 cubic feet of fluid is in the shape of a cylinder with a spherical cap top. The cylindrical shell part costs \$34 per square foot, the spherical cap part costs \$58 per square foot, and the floor of the container costs \$41 per square foot. How should the container be designed so that the costs are minimized?

↑ from review questions.

4. Find all critical points of the functions below, and classify the types (max, min, saddle) of critical points.

(a) $A(x, y) = 45y - 9x^2 - 12y^2 + y^3 + 3x^2 y - 52$

(b) $B(s, t) = (s^2 - 2t)\, e^{s+t}$

(b) $\nabla B = (B_s, B_t)$

$= \left(2s e^{s+t} + (s^2-2t)e^{s+t},\ -2e^{s+t} + (s^2-2t)e^{s+t}\right)$

$= (0,0)$

$$(2s + s^2 - 2t)\cancel{e^{s+t}} = 0$$

$$(-2 + s^2 - 2t)\cancel{e^{s+t}} = 0$$

$$\Rightarrow s^2 - 2t = 2$$

$$(2s + 2) = 0$$

$$2(s+1) = 0$$

$$s = -1$$

$$2(-1) + 1 - 2t = 0$$

$$-1 - 2t = 0$$

$$2t = -1$$

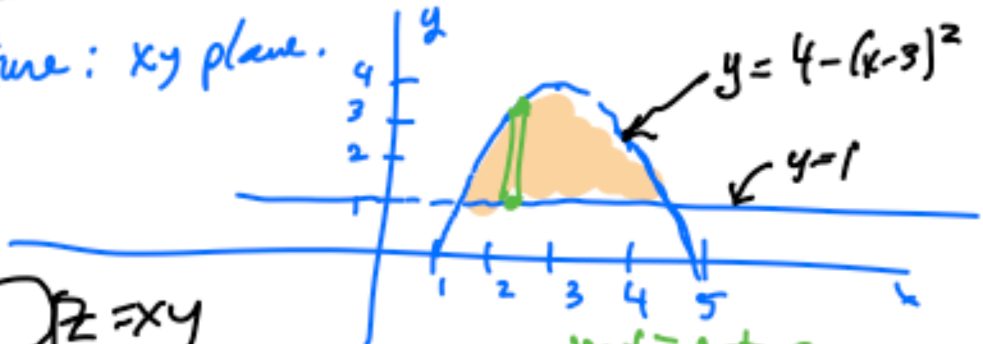
$$t = -\frac{1}{2}$$

From last time:

<u>integrals</u>

Example Find the volume under $z = xy + 20$ over the region in the xy plane between $y = 1$ and $y = 4 - (x-3)^2$.

Picture: xy plane.



vertical strip

$$\text{Volume} = \int_{x=}^{x=} \left(\int_{y=1}^{y=4-(x-3)^2} xy+20 \, dy \right) dx$$

intersection points: $4-(x-3)^2 = 1$

$$3 = (x-3)^2$$

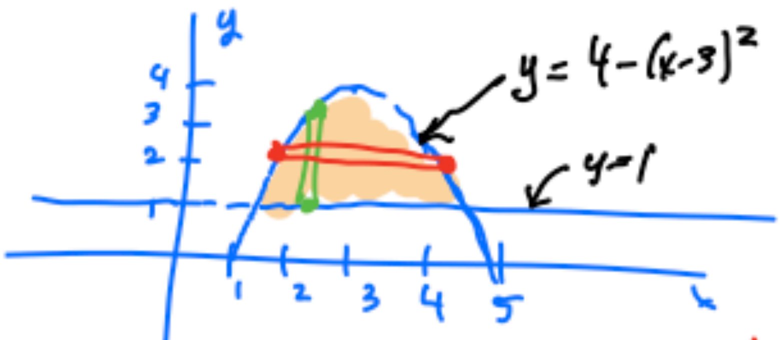
$$\pm\sqrt{3} = x - 3$$

$$x = 3 \pm \sqrt{3}.$$

$$\text{Volume} = \int_{x=3-\sqrt{3}}^{x=3+\sqrt{3}} \left(\int_{y=1}^{4-(x-3)^2} xy+20 \, dy \right) dx$$

$$= \boxed{\frac{532}{5}\sqrt{3}} \, . \quad \text{(see sagemath)}$$

What if we used horizontal strips first?



y constant

$$y = 4-(x-3)^2$$

$$(x-3)^2 = 4-y$$

$$x-3 = \pm\sqrt{4-y}$$

$$x = 3 \pm \sqrt{4-y}$$

$$\int (xy+20)\, dx$$

$$\int_{y=1}^{4} \left(\int_{x=3-\sqrt{4-y}}^{3+\sqrt{4-y}} (xy+20)\, dx \right) dy$$

$$\underset{\text{Sagemath}}{=} \frac{532\sqrt{3}}{5}$$

Same answer!